

Four loop twist two, BFKL, wrapping and strings

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December 16, 2008

Abstract

The anomalous dimensions of twist two operators have to satisfy certain consistency requirements derived from BFKL. For $\mathcal{N} = 4$ SYM it was shown that at four loops, the anomalous dimensions derived from the all-loop asymptotic Bethe ansatz do not pass this test. In this paper we obtain the remaining wrapping part of these anomalous dimensions from string theory and show that these contributions exactly cure the problem and lead to agreement with both LO and NLO BFKL expectations.

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1 Introduction

The integrable structures which appear both on the gauge theory side [1, 2, 3, 4, 5] and the string theory side [6] of the AdS/CFT correspondence [7] give hope to find the full spectrum of both theories. On the gauge theory side we are interested in finding the anomalous dimensions of gauge invariant operators in $\mathcal{N} = 4$ super Yang-Mills theory, while on the string theory side we would like to determine the energies of the superstring excitations in the $AdS_5 \times S^5$ background.

A lot of progress has been done for specific type of operators, namely with large numbers of fields, and for the corresponding infinitely long/fast strings (strings with large charges). The S-matrix for elementary excitations has been predicted from the symmetry algebra up to the overall scalar factor [8] which was finally fixed in [9, 10] using the crossing symmetry [11]. The Bethe Ansatz Equations have been derived [3, 12, 13, 14, 15, 16, 17, 18] giving the spectrum of states with large quantum numbers.

Nevertheless, it is known that the Bethe Ansatz is not valid for short operators and strings with small charges. On the gauge theory side it is due to the fact that at the order of λ^L , where L is the length of the gauge invariant operator, wrapping corrections start to play a role. In an analogous manner [19], on the string theory side virtual corrections around the worldsheet cylinder appear. In [20] the leading exponential finite size effects at strong coupling for a single giant magnon were computed from the identification with Lüscher-like corrections (see also [21, 22, 23]) and turned out to agree exactly with the expression obtained from classical string solution [24]. At weak coupling, wrapping corrections were analyzed from the gauge theory perturbative point of view in [25]. The wrapping contribution to the Konishi operator anomalous dimension at four loops has been explicitly computed within perturbative gauge theory [26] (a subsequent independent ab-initio perturbative computation of [27] which included both the wrapping and non-wrapping graphs reaffirmed this result). The same result was obtained purely within string theory from the $AdS_5 \times S^5$ string sigma model in [28].

The wrapping effects appear in the most manifest way for the shortest possible operator. The simplest non protected operators, which are generalizations of the Konishi operator, are twist two operators: $\text{tr } Z D^{2M} Z$. Apart from being a natural testing ground for wrapping effects, which should appear at 4 loops, their anomalous dimensions are also interesting for their

own sake as they are intertwined with various seemingly unrelated physical processes and have rich analytical properties. Moreover these operators (in fact their very close analogues) are also of big importance in ordinary QCD (see in particular [29] in the context of the present paper).

Their anomalous dimensions have been computed perturbatively (in QCD) up to 3 loops [30] and using the maximum transcendentality conjecture [31] for $\mathcal{N} = 4$ SYM the corresponding answer in $\mathcal{N} = 4$ SYM has been extracted [32, 33, 34]. This has been found to agree with the prediction from the asymptotic Bethe Ansatz. Subsequently the Bethe Ansatz answer for four loops was computed [35] and found to satisfy the maximal transcendentality principle. It was, however, demonstrated that the Bethe Ansatz answer is in conflict with predictions from the BFKL equation [36]. It was pointed out that the missing part of the four-loop answer should come from wrapping effects. The motivation of our paper is to fill this gap and find the leading wrapping correction for general twist two operators and reexamine the consistency with BFKL.

Thus our aim is to extend the calculations of the wrapping corrections to the case of arbitrary twist two operators. We will adapt the method used in the case of Konishi operator in [28], based on the integrability properties of the worldsheet quantum field theory of the superstrings in $AdS_5 \times S^5$. The leading contribution from wrapping will appear at order λ^4 which exactly matches the predictions coming from TBA type considerations [19].

The operators considered in this paper are part of a larger family of interesting operators from the $\mathfrak{sl}(2)$ sector made up from M derivatives (spin) and J complex scalars (twist): $\text{tr}(D^{s_1} Z \dots D^{s_J} Z)$. For high er twist, there is also a wide range of interesting phenomena e.g. the large spin (M) and twist (J) behaviour of their anomalous dimension $\Delta(M, J)$ is determined by the BES/FRS integral equations [10, 37]. Remarkably, in the limit when both $M, J \rightarrow \infty$ such that $j = \frac{J}{\log M}$ is kept fixed the leading logarithmic behaviour is governed by the cusp anomalous dimension and the ground-state energy density of the $O(6)$ σ model [38, 39, 40, 41, 42, 43, 44, 45, 46, 47]. We expect that the methods presented in the present paper should also be applicable for the higher twist case.

The plan of this paper is as follows. In section 2 we briefly introduce the twist two operators and their main properties. In section 3 the formula for wrapping correction for general twist two operator is given. In section 4 we point out all ingredients needed to compute the leading wrapping correction. All relevant calculations is presented in section 5, while the final answer is

obtained in section 6. Section 7 shows that the wrapping corrections do not change the large M asymptotic behaviour of anomalous dimension for twist two operators, consequently, the cusp anomalous dimension (‘scaling function’) is untouched by wrapping. In section 8 we analytically continue the final result to $M = -1 + \omega$ and then compare it with the BFKL prediction. We close the paper with a discussion and several appendices.

2 Twist two operators and BFKL

Twist two operators in $\mathcal{N} = 4$ SYM are operators which are made from two Z fields ($J = 2$) and an arbitrary (but even) number of derivatives M in a fixed light cone direction. Schematically these operators are thus of the form $\text{tr } Z D^M Z$. For each even M there is a unique highest weight non BPS operator and so we may define the anomalous dimension

$$\Delta(M, J = 2) = 2 + M + \sum_{l=1}^{\infty} \gamma_{2l}(M) g^{2l} \quad (1)$$

where $g^2 = \lambda/16\pi^2$. Here the index l denotes the perturbative loop order. This quantity can be computed from the asymptotic Bethe ansatz in the $\mathfrak{sl}(2)$ sector exactly up to 3 loops. The answer from the Bethe ansatz at 4 loops and higher will have to be supplanted by the contribution of so-called ‘wrapping interactions’. The aim of this paper is to compute this contribution at 4 loop order from the string sigma model in $AdS_5 \times S^5$. This arises due to the identification of the anomalous dimensions with energies of string states in $AdS_5 \times S^5$. Since the worldsheet QFT is integrable, and we know the exact S-matrix, we know the full on-shell data of the worldsheet QFT at infinite volume. We may therefore study the spectrum of energies around the infinite volume limit. The leading piece is contained in the Bethe Ansatz (identical to the asymptotic Bethe ansatz), while the leading virtual corrections computed from generalized multiparticle Lüscher formulas provide a way to compute *exactly* the 4-loop wrapping corrections. Thus we may split the 4-loop coefficient of (1) into

$$\gamma_8(M) = \gamma_8^{Bethe}(M) + \gamma_8^{wrapping}(M) \quad (2)$$

The Bethe ansatz term has been computed in [35] and can be found in table 1 of that reference. The aim of this paper is to compute the wrapping

part $\gamma_8^{wrapping}(M)$ from multiparticle Lüscher formulas for the $AdS_5 \times S^5$ worldsheet QFT. This is a generalization of the computation of the 4-loop anomalous dimension of the Konishi operator [28] which corresponds to

$$\gamma_8^{wrapping}(2) = 324 + 864\zeta(3) - 1440\zeta(5) \quad (3)$$

The function $\Delta(M, J)$ has numerous interesting properties. Firstly, its large M limit is related to the cusp anomalous dimension [48]:

$$\lim_{M \rightarrow \infty} \Delta(M, J) - J - M \sim 2\gamma_{cusp}(g) \log M \quad (4)$$

It is therefore expected that the coefficient of $\log M$ does not depend on J . The cusp anomalous dimension can be investigated both from the perturbative side [49] and from the strong coupling side [50, 51, 40, 52, 53, 54, 55] with an interpolating answer following from the BES equation derived from the asymptotic Bethe ansatz [10]. The applicability of the Bethe ansatz analysis rests on the independence of $\gamma_{cusp}(g)$ on J so that wrapping interactions do not contribute in this limit. This can be argued both on the perturbative side [56] and on the strong coupling string side [57], but it would be interesting to verify directly that wrapping contributions will vanish in this limit. This is among the aims of the present paper where we verify this for $J = 2$ and 4-loop order. Once these wrapping corrections are obtained, thus completing our knowledge of 4 loop anomalous dimensions of twist two operators, there are also other very interesting features of their large M limit which could be investigated in particular its ‘reciprocity’ properties [58], further studied in [59].

Secondly, the coefficients of (1) at l -loop, $\gamma_{2l}(M)$ are expected to obey, for $\mathcal{N} = 4$ SYM, the principle of maximum transcendentality [31]. This means that they are expressed in terms of (nested) harmonic sums and ζ functions such that the degree of transcendentality of $\gamma_{2l}(M)$ equals $2M - 1$. The degree of transcendentality for a product is defined to be the sum of the degrees of each factor. The degree of transcendentality of $\zeta(n)$ is defined to be n while the degree of the harmonic sum $S_{i_1, i_2, \dots, i_k}(M)$ is $\sum_i |i_k|$. The harmonic sum $S_{i_1, i_2, \dots, i_k}(M)$ is defined recursively as

$$S_{i_1, i_2, \dots, i_k}(M) = \sum_{n=1}^M \frac{\text{sign}(i_1)^n}{n^{|i_1|}} S_{i_2, \dots, i_k}(n) \quad (5)$$

while the elementary harmonic sums with a single index are given by

$$S_j(M) = \sum_{n=1}^M \frac{\text{sign}(j)^n}{n^{|j|}} \quad (6)$$

In [35] it was verified that the part of the 4-loop result following from the asymptotic Bethe ansatz indeed is composed of terms of transcendentality degree 7. We would like to verify in the present paper that the contribution of wrapping corrections will also have the same degree of transcendentality.

The third and perhaps the most nontrivial property of the anomalous dimensions (1) is the interrelation with BFKL [36] and NLO BFKL [60] equations which describe perturbatively gauge theory high energy scattering in the Regge limit. From this formalism one obtains a specific prescription for the pole structure of the *analytical continuation* of $\Delta(M, J=2)$ to $M = -1 + \omega$. The l -loop coefficients of (1) have to have the following pole structure around $M = -1 + \omega$ [35]:

$$\gamma_2(\omega) \sim -4 \left(\frac{2}{\omega} + 0 + \mathcal{O}(\omega) \right) \quad (7)$$

$$\gamma_4(\omega) \sim -16 \left(\frac{0}{\omega^2} + \frac{0}{\omega} + \mathcal{O}(\omega^0) \right) \quad (8)$$

$$\gamma_6(\omega) \sim -64 \left(\frac{0}{\omega^3} + \frac{\zeta(3)}{\omega^2} + \mathcal{O}\left(\frac{1}{\omega}\right) \right) \quad (9)$$

$$\gamma_8(\omega) \sim -256 \left(\frac{4\zeta(3)}{\omega^4} + \frac{\frac{5}{4}\zeta(4)}{\omega^3} + \mathcal{O}\left(\frac{1}{\omega^2}\right) \right) \quad (10)$$

where the leading pole comes from the BFKL equation while the second one is a consequence of NLO BFKL. It has been verified [35] that the 1-, 2- and 3-loop result exactly agrees with the above BFKL and NLO BFKL predictions. The main conclusion of [35] was that the Bethe ansatz part of $\gamma_8(\omega)$ does not satisfy this constraint. Its expansion around $M = -1 + \omega$ is¹

$$\gamma_8^{Bethe}(\omega) \sim 256 \left(\frac{-2}{\omega^7} + \frac{0}{\omega^6} + \frac{8\zeta(2)}{\omega^5} - \frac{13\zeta(3)}{\omega^4} - \frac{16\zeta(4)}{\omega^3} + \mathcal{O}\left(\frac{1}{\omega^2}\right) \right) \quad (11)$$

The main motivation of this work is to compute the corresponding wrapping contribution $\gamma_8^{wrapping}(M)$ and to check whether the sum $\gamma_8(\omega) = \gamma_8^{Bethe}(\omega) +$

¹We are grateful to Vitaly Velizhanin and the authors of [35] for informing us of this explicit expansion.

$\gamma_8^{wrapping}(\omega)$ has the correct analytical properties required by LO and NLO BFKL.

3 Wrapping correction for twist two operators

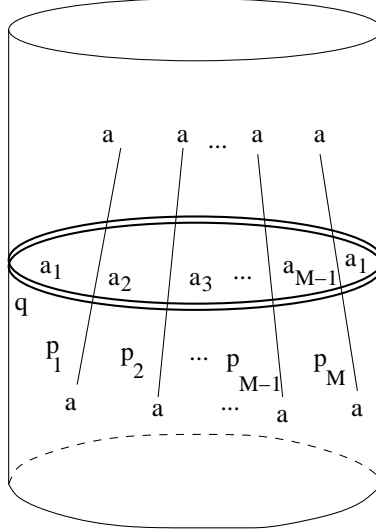


Figure 1: Multiparticle Lüscher correction. The vertical lines represent the physical particles forming the multiparticle state, while the double line loop represents the on-shell ‘virtual’ particle with complex momentum.

The aim of this section is to extend the calculation of the leading wrapping correction of the Konishi operator to generic twist two operators. This amounts to calculating the leading finite size energy correction of a specific M particle state. In [28] the authors conjectured a Lüscher type formula for generic multiparticle states and successfully applied it for the Konishi case: $M = 2$. The formula consists of two parts: The first describes the effect how the finite volume modifies the particles’ quantization conditions and shifts their momenta, while the second is due to virtual particles propagating around the cylinder and directly changes the energy as

$$\Delta E(L) = - \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \text{STr}_{a_1} [S_{a_1 a}^{a_2 a}(q, p_1) S_{a_2 a}^{a_3 a}(q, p_2) \dots S_{a_M a}^{a_1 a}(q, p_M)] e^{-\tilde{\epsilon}_{a_1}(q)L} \quad (12)$$

The content of this formula is schematically represented in Fig. 1. This formula applies for an M particle state with all particles of type a , such that their consecutive self-scatterings preserve this state and determine their momenta p_i by the BA equations. The matrix $S_{ba}^{ca}(q, p)$ describes how a virtual particle of type b with momentum q scatters on the real particle of type a and momentum p . The exponential factor can be interpreted as the propagator of the virtual particle. What makes difficult to apply this formula in practice is that we have to sum over all possible virtual particles in the theory (both fundamental magnons $Q = 1$ and the infinite tower of their bound states $Q > 1$), their polarizations a_1 and, even more, over all possible intermediate states a_2, \dots, a_M .

Let us focus on the leading wrapping correction for the anomalous dimension of a twist two operator. Since the modification of the BA equation appears at order g^8 its contribution to the energy will be subleading and it is sufficient to analyze equation (12). In the next section we explain all the ingredients of this equation specified to the twist two case, while in the subsequent one we perform the actual calculation.

4 Main ingredients of the wrapping correction

In order to apply formula (12) for twist two operators we have to explain the following ingredients: What is the state such an operator corresponds to and how are their momenta determined by the BA equation? What are the virtual excitations, what is their exponential damping factor and how they scatter on the state that corresponds to the twist two operator? Let us investigate these questions in this order.

Multiparticle states corresponding to twist two operators

The state which corresponds to twist two operators is an M particle state with M even. According to the $su(2|2) \otimes su(2|2)$ classification each particle belong to the $\mathfrak{sl}(2)$ sector of the fundamental representation, which is realized in terms of the totally anti-symmetric representation, thus has label

$a = (1, i)^2$. The volume is $L = 2$ and the momenta of the particles are determined by the BA equation. At leading order in g their rapidities

$$u(p) = \frac{1}{2} \cot \frac{p}{2} \sqrt{1 + 16g^2 \sin^2 \frac{p}{2}}$$

are given as the roots of Baxter's Q function, $P_M(u)$. This is a polynomial of order M which is given explicitly (at 1-loop level³) by the generalized hypergeometric function (see [62, 63] and the methods of [64])

$$P_M(u) = {}_3F_2(-M, M+1, \frac{1}{2} - iu; 1, 1|1)$$

Since this polynomial is even each rapidity u_k comes in pairs:

$$P_M(u) \propto \prod_{k=1}^M (u - u_k) = (u - u_1)(u + u_1) \dots (u - u_{\frac{M}{2}})(u + u_{\frac{M}{2}}) \quad (13)$$

This is a useful form that we will use later on.

Virtual particles and the exponential factor

The virtual particles turn out to belong to the completely anti-symmetric representation of $su(2|2) \otimes su(2|2)$ first discussed in [65]. This representation exists for any integer Q and has dimension $(4Q)^2$. The dispersion relation of their particles leads at leading order to the exponential factor

$$e^{-\tilde{\epsilon}_Q(q)L} = e^{-2L \operatorname{arcsinh} \frac{\sqrt{q^2 + Q^2}}{4g}} = \left(\frac{z^-}{z^+} \right)^L = \frac{4^L g^{2L}}{(q^2 + Q^2)^L}$$

where here and later on we use that

$$z^\pm(q) = \frac{q + iQ}{4g} \left(\pm 1 + \sqrt{1 + \frac{16g^2}{q^2 + Q^2}} \right)$$

Observe that only z^+ scales like g^{-1} but z^- goes as g in the weakly coupled regime.

²We use the convention in which fermionic coordinates have labels 1, 2 while bosonic ones 3, 4. See [28] for details.

³This expression has been generalized to higher loops in [61]. However we will not need these expressions for our computation as they would contribute only at higher loop orders.

The scattering matrix

In [28] it was described how a particle with charge Q and parameters z^\pm scatters on a fundamental particle with $Q = 1$ and parameters x^\pm

$$x^\pm(u) = \frac{2u \pm i}{4g} \left(1 + \sqrt{1 - \frac{16g^2}{(2u \pm i)^2}} \right)$$

The scattering matrix can be determined, using the superfield methods of [66], from the requirement that it commutes with the symmetry charges of $su(2|2) \otimes su(2|2)$ up to a scalar factor which was obtained from the bootstrap. Let us recall the scalar and matrix part of the scattering matrix.

Scalar part

In the bootstrap procedure the composite particle z^\pm is realized in terms of its individual constituents z_i^\pm such that the bound-state condition is realized: $(z^- = z_1^-, z_1^+)$, $(z_2^- = z_1^+, z_2^+ = z_3^-)$, \dots , $(z_Q^- = z_{Q-1}^+, z_Q^+ = z^+)$. The scalar factor of the $Q-1$ scattering is then nothing but the product of the individual scalar factors of the $1-1$ scatterings:

$$S_{Q-1}^{sl(2)}(z^\pm, x^\pm) = \prod_{i=1}^Q S_{1-1}^{sl(2)}(z_i^\pm, x^\pm) \quad (14)$$

where

$$S_{1-1}^{sl(2)}(z^\pm, x^\pm) = \frac{z^- - x^+}{z^+ - x^-} \frac{1 - \frac{1}{z^+ x^-}}{1 - \frac{1}{z^- x^+}} \quad (15)$$

The choice for the constituents is not unique and such configuration was adopted which had the most z_i^\pm parameters of order g^{-1} . This leads to the result which, in terms of q and u , can be written as:

$$S_{Q-1}^{sl(2)}(q, u) = \frac{(q - i(Q-1) - 2u)(2u + i)^2}{(q - i(Q+1) - 2u)(q + i(Q-1) - 2u)(q + i(Q+1) - 2u)}$$

Matrix part

In describing the matrix part we can exploit the fact that it is a tensor product of two copies of the same S-matrices. This reflects the $su(2|2) \otimes su(2|2)$ nature

of the symmetry. Thus it is enough to analyze one copy of the anti-symmetric representations. It has $4Q$ states: $2Q$ bosons and $2Q$ fermions. From the $su(2|2)$ symmetry it is possible to calculate how these particles scatter on the particles of the fundamental representation, see [28] for the result.

For calculating the wrapping correction we need only the S_{II}^{1J} , $I, J = 1, \dots, 4Q$ part of the matrix. We will interpret it as a matrix acting on the anti-symmetric representation of charge Q and denote it by $S_{Qmatrix}^{sl(2)}(u, q)$. Lets us go through systematically all possible matrix elements. We use the notation of [28] adapted to anti-symmetric representations. The first bosonic state $I = 1$ can scatter into itself

$$SB0(q, u) := S_{11}^{11}(q, u)$$

The bosonic state $I = j$, $j = 2, \dots, Q$ can scatter either into itself, or into the other bosonic state $J = j + Q$. In a similar manner this other bosonic state with $J = j + Q$ can either scatter into itself or scatter back to $I = j$. Thus for each $j = 2, \dots, Q$ these scatterings form a 2×2 matrix

$$SB(q, u, j+1) = \begin{pmatrix} S_{1j}^{1j}(q, u, j) & S_{1j}^{1j+Q}(q, u, j) \\ S_{1j+Q}^{1j}(q, u, j) & S_{1j+Q}^{1j+Q}(q, u, j) \end{pmatrix}$$

The boson with index $I = Q + 1$ scatters again into itself

$$SBQ(q, u) = S_{1Q+1}^{1Q+1}(q, u)$$

The fermionic part of the scattering is much simpler, since it is diagonal and is the same for both fermionic particles:

$$SF(q, u, j+1) = S_{12Q+j}^{12Q+j}(q, u) = S_{13Q+j}^{13Q+j}(q, u)$$

The explicit form of these matrix elements can be found in the Appendix of [28]. Here, for convenience, we grouped them slightly differently by shifting some of the labels. (For the paper to be self contained, we list the needed matrix elements explicitly in Appendix D).

5 The calculation of the wrapping correction

Having introduced all the needed quantities we can formulate the wrapping correction as

$$\Delta E = - \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \prod_{k=1}^M S_{Q-1}^{sl(2)}(q, u_k) \left(\text{STr} \left(\prod_{k=1}^M S_{Qmatrix}^{sl(2)}(q, u_k) \right) \right)^2 \frac{16g^4}{(q^2 + Q^2)^2}$$

Where the super-trace part can be further elaborated as

$$\begin{aligned} \text{STr}\left(\prod_{k=1}^M S_{Q_{matrix}}^{sl(2)}(q, u_k)\right) &= \prod_{k=1}^M SB0(q, u_k) + \prod_{k=1}^M SBQ(q, u_k) \\ &\quad - 2 \sum_{j=0}^{Q-1} \prod_{k=1}^M SF(q, u_k, j) + \sum_{j=1}^{Q-1} \text{Tr} \left(\prod_{k=1}^M SB(q, u_k, j) \right) \end{aligned} \quad (16)$$

We evaluate this part carefully in Appendix A and here we just summarize the main steps. Since we expect the wrapping contribution to appear at order g^8 , and there is an explicit g^4 dependence coming from the ‘exponential’ part, the super-trace part has to be of order g^2 , so its leading term should vanish. We check this requirement first and then make a systematical small g^2 expansion of each term later. Schematically we expand each function as $f(g) = f_0(1 + g^2 \delta f + O(g^4))$. The key feature why we were able to do the calculation is, that the leading order part of the bosonic matrices $SB(q, u_k, j)_0$ can be diagonalized in a u_k -independent way. This enables us to rewrite the bosonic matrix contribution as a sum over the contributions of the two eigenvalues $SB^\pm(q, u_k, j)_0$. Explicitly we found that

$$SB^+(q, u_k, j)_0 = \frac{i + 2ij + q - iQ - 2u}{i + q - iQ - 2u}$$

Additionally, when we extended the summation for these eigenvalues (from 1 to 0 for + and from $Q-1$ to Q for -) we could incorporate the contributions of the two separate bosonic terms $SB0(q, u_k)_0$ and $SBQ(q, u_k)_0$. We also observed that $SB^-(q, u_k, j+1) = SB^+(q, u_k, j) \frac{2u+i}{2u-i}$ such that the - bosonic summation can be also shifted and the zero order part for wrapping has the form:

$$\text{STr}\left(\prod_{k=1}^M S_{Q_{matrix}}^{sl(2)}(q, u_k)\right)_0 = -2 \sum_{j=0}^{Q-1} \prod_{k=1}^M SF(q, u_k, j)_0 + 2 \sum_{j=0}^{Q-1} \prod_{k=1}^M SB^+(q, u_k, j)_0$$

Further checking that the fermionic parts and the bosonic parts are the same for each j , ($SF(q, u_k, j)_0 = SB^+(q, u_k, j) \sqrt{\frac{2u+i}{2u-i}}$), we could see that they completely annihilate each other at this zeroth order.

In doing the calculation at first order we performed the same steps: We diagonalized the bosonic contributions upto $O(g^2)$, we extended the bosonic

summation to incorporate the extra separate bosonic pieces, shifted the summation for the second bosonic part, and finally exploited the fact that the fermionic and bosonic zero order terms are the same for any j . As a result we arrived at formula

$$\text{STr}\left(\prod_{k=1}^M S_{Qmatrix}^{sl(2)}(q, u_k)\right) = g^2 \sum_{j=0}^{Q-1} \left(\prod_{k=1}^M SB^+(q, u_k, j)_0\right) \sum_{k=1}^M \delta SBF(q, u_k, j)$$

where after some lengthy but straightforward calculations we found that

$$\delta SBF(q, u_k, j) = \frac{16}{1 + 4u_k^2} \left[\frac{1}{2j - iq - Q} - \frac{1}{2(j+1) - iq - Q} \right]$$

Exploiting the very simple u_k dependence of the summand we can recognize the one loop BA energy and replace it with the harmonic sum $S_1(M)$ as

$$\sum_{k=1}^M \frac{16}{1 + 4u_k^2} = 8S_1(M) = 8 \sum_{l=1}^M \frac{1}{l}$$

In order to abbreviate future formulas we will suppress the arguments of harmonic sums if they are M . The equation (13) can be used to write the result in a more economical way:

$$\prod_{k=1}^M SB^+(q, u_k, j)_0 = \frac{P_M(\frac{1}{2}(q - i(Q-1) + ij))}{P_M(\frac{1}{2}(q - i(Q-1)))}$$

The final form of the wrapping correction

We arrived at one of the main results of the paper. We merely have to collect all the ingredients of the wrapping correction: the exponential factor, the scalar factor and the matrix part. There is an elegant way of writing the whole wrapping correction in terms of Baxter's Q function⁴ as

$$\Delta E = -64g^8 S_1^2 \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{T_M(q, Q)^2}{R_M(q, Q)} \frac{16}{(q^2 + Q^2)^2} \quad (17)$$

⁴We denote the Baxter's function as $P_M(\cdot)$ in order to avoid confusion with the Q parameter.

where

$$R_M(q, Q) = P_M\left(\frac{1}{2}(q-i(Q-1))\right)P_M\left(\frac{1}{2}(q+i(Q-1))\right)P_M\left(\frac{1}{2}(q+i(Q+1))\right)P_M\left(\frac{1}{2}(q-i(Q+1))\right)$$

and

$$T_M(q, Q) = \sum_{j=0}^{Q-1} \left[\frac{1}{2j-iq-Q} - \frac{1}{2(j+1)-iq-Q} \right] P_M\left(\frac{1}{2}(q-i(Q-1)) + ij\right)$$

An alternative way of writing this function is

$$T_M(q, Q) = \frac{iP_M\left(\frac{1}{2}(q-i(Q-1))\right)}{q-iQ} - \frac{iP_M\left(\frac{1}{2}(q+i(Q-1))\right)}{q+iQ} + \tilde{T}_M(q, Q)$$

where

$$\tilde{T}_M(q, Q) = \sum_{j=1}^{Q-1} \frac{P_M\left(\frac{1}{2}(q-i(Q-1)) + ij\right) - P_M\left(\frac{1}{2}(q-i(Q+1)) + ij\right)}{2j-iq-Q}$$

It is not difficult to show that $\tilde{T}_M(q, Q)$ defines a polynomial.

Wrapping correction for odd particle number

In computing the anomalous dimension of twist two operators the consideration of odd states played an important role [35]. This meant an analytical continuation of the result from even to odd M . The hypergeometric function determines the allowed rapidities: they are all paired except one which has $u = 0$. This is not physical since it corresponds to $p = \pi$ and not $p = 0$. Nevertheless, we can try to extend the wrapping correction to this case. The first problem immediately arises at zeroth order. The extra $p = \pi$ state gives different extra factors to the two bosonic and fermionic contributions. Namely the B^+ part is unchanged, the B^- gets an additional -1 and annihilates B^+ , while both fermions got an extra i . Since we would like to maintain for the analytical continuation that the correction starts at g^8 we suppose that in a theory where $u = 0$ is an allowed state the symmetry between the two fermions are broken and their contributions have opposite signs. Thus we add extra factors as i and $-i$ to the two fermionic components. With this assumption we could calculate the wrapping correction of those odd states.

The only difference compared to the even one resides in the function $T(q, Q)$ which turns out to be

$$T_M^{odd}(q, Q) = \sum_{j=0}^{Q-1} \left[\frac{1}{2j - iq - Q} + \frac{1}{2(j+1) - iq - Q} \right] P_M \left(\frac{1}{2}(q - i(Q-1)) + ij \right)$$

So in essence a ‘nice’ analytical continuation of (17) amounts to inserting an additional $(-1)^M$ sign in the definition of $T_M(q, Q)$.

Calculating the integral

We calculate the integral by closing the contour on the upper half plane and taking residues over the poles of the integrand. We have two types of poles. The pole at $q = iQ$ is of kinematical origin since it does not depend on the scattering matrix. In contrast, the poles of the function $R(q, Q)$ depends on u_k thus are determined by the dynamics and come from the scattering matrix. They correspond to μ terms in the Lüscher correction. In the paper [28] an argument, based on kinematical considerations, was given that such terms are absent in the weakly coupled ($g \rightarrow 0$) limit, so we expect their contribution to vanish. Although they do not vanish separately but when we sum up their contributions for all the bound-states (for Q) the result is indeed zero. We checked this explicitly for the first few M cases. As a consequence in calculating the integral for the wrapping correction we can take the residue only at the kinematical pole $q = iQ$. Subsequently we have to sum the resulting expression for Q running from 1 to ∞ .

6 Determination of the final form of $\gamma_8^{wrapping}(M)$

Assuming the maximum transcendentality principle, we expect that $\gamma_8^{wrapping}(M)$ has the following structure:

$$\gamma_8^{wrapping}(M) = C_7(M) + C_4(M)\zeta(3) + C_2(M)\zeta(5) \quad (18)$$

where the coefficients $C_n(M)$ have transcendentality degree n .

In Appendix B we have analytically derived the coefficients of $\zeta(3)$ and $\zeta(5)$ and found that indeed they have the expected transcendentality structure⁵:

$$C_2(M) = -640S_1^2 \quad C_4(M) = -512S_1^2S_{-2} \quad (19)$$

⁵In the following all harmonic sums are evaluated at M so we suppress the argument.

It remains to determine the rational part $C_7(M)$. A-priori the number of independent harmonic sums of transcendentality 7 is quite large, but we may use the structure of the Lüscher correction i.e. the fact that S_1^2 gets automatically factored out to significantly simplify the analysis. Hence we may write $C_7(M) = S_1^2 C_5(M)$. We are thus left with harmonic sums of transcendentality 5. Assuming that the index -1 does not appear we are left with a basis of 41 harmonic sums.

We then computed analytically the residues of the integrand at $q = iQ$ and summed up the resulting expression from $Q = 1$ to ∞ . We did this for $M = 1$ to $M = 41$, where we included, similarly as in [35], also the unphysical odd values of M . A justification of the precise form of analytical continuation of the even M integrands to odd M is the agreement of the coefficients of $\zeta(3)$ and $\zeta(5)$ and the fact that the remaining term was just a rational number. The results from $M = 1$ to $M = 41$ fixed all the coefficients of the 41 harmonic sums which turned out to be simple integers. The result for the rational part obtained in this way is

$$C_7(M) = 256S_1^2 (-S_5 + S_{-5} + 2S_{4,1} - 2S_{3,-2} + 2S_{-2,-3} - 4S_{-2,-2,1}) \quad (20)$$

As a check we then computed the wrapping correction from Lüscher formulas for $M = 42$ and $M = 44$ and got perfect agreement with the above formulas (18)-(20).

In the remaining part of the paper we will first verify that $\gamma_8^{wrapping}(M)$ does not have any contribution of the order of $\log M$ in the large M limit so that the cusp anomalous dimension remains unchanged. Then we will analyze the analytical continuation of $\gamma_8^{wrapping}(M)$ to $M = -1 + \omega$ and check compatibility with LO and NLO BFKL.

7 Large M asymptotics of $\gamma_8^{wrapping}(M)$

Wrapping corrections should not change the leading asymptotic behaviour of the anomalous dimension of twist-2 operators. On the other hand the final formula which we obtained is of the form

$$\gamma_8^{wrapping}(M) = 128S_1^2(M) \times (\text{finite when } M \rightarrow \infty)$$

where $S_1^2(M)$ scales as $\log^2(M)$ when $M \rightarrow \infty$. The only option for the wrapping correction not to change the leading large M asymptotics is that

the finite part vanishes at infinity. If this is the case then the first wrapping contribution to the asymptotic behaviour may enter at the order $\frac{\log^2(M)}{M}$ which is subleading comparing with $\log(M)$.

Let us check if the leading expansion around $M = +\infty$ of our result vanishes. The values at infinity of the nested harmonic sums can be expressed in terms of multivariate zeta functions (Zagier-Euler sums) as was shown in [67]. These sums can be reexpressed in terms of ordinary Euler sums. The relations between the former and the later can be found using EZ-Face — an on-line calculator for Euler sums [68]. The results relevant for us are

$$\begin{aligned}
S_{-2}(\infty) &= -\frac{1}{2}\zeta(2) \\
S_{-5}(\infty) &= -\frac{15}{16}\zeta(5) \\
S_5(\infty) &= \zeta(5) \\
S_{4,1}(\infty) &= -\zeta(2)\zeta(3) + 3\zeta(5) \\
S_{3,-2}(\infty) &= \frac{1}{4}\zeta(2)\zeta(3) - \frac{51}{32}\zeta(5) \\
S_{-2,-3}(\infty) &= \frac{21}{8}\zeta(2)\zeta(3) - \frac{67}{16}\zeta(5) \\
S_{-2,-2,1}(\infty) &= \frac{15}{16}\zeta(2)\zeta(3) - \frac{29}{32}\zeta(5)
\end{aligned}$$

Plugging them into the coefficient of S_1^2 in the wrapping correction

$$-5\zeta(5) - 4S_{-2}\zeta(3) - 2S_5 + 2S_{-5} + 4S_{4,1} - 4S_{3,-2} + 4S_{-2,-3} - 8S_{-2,-2,1}$$

we obtain that it vanishes as we expected. It would be very interesting to analyze the large M limit in more detail along the lines of [58, 59].

8 Analytical continuation of $\gamma_8^{wrapping}(M)$

The way how harmonic sums can be analytically continued was nicely explained in [67]. There is a pronounced difference between the cases when all indexes are positive and when we have at least one negative index. The former case can be described systematically, while the latter one requires a case by case study. We relegate the details of the analytical continuation of the needed harmonic sums to Appendix C. Using the results there we can

list the analytical continuation of all harmonic sums of interest. Since we are concerned with the singularity around -1 only up to the third order pole, it is enough to expand S_1 to third order and keep only the singular part of all other harmonic sums:

$$\begin{aligned}
S_1(-1 + \omega) &= -\frac{1}{\omega} + \omega\zeta(2) - \omega^2\zeta(3) + \omega^3\zeta(4) + \dots \\
S_5(-1 + \omega) &= -\frac{1}{\omega^5} + \dots \\
S_{4,1}(-1 + \omega) &= -\frac{1}{\omega^4}(\zeta(2)\omega - \zeta(3)\omega^2 + \zeta(4)\omega^3 + \dots) + \dots \\
S_{-5}(-1 + \omega) &= \frac{1}{\omega^5} + \dots \\
S_{3,-2}(-1 + \omega) &= -\frac{1}{\omega^3}(2\zeta(-2) - 2\zeta(-3)\omega + 3\zeta(-4)\omega^2 + \dots) + \dots \\
S_{-2,-3}(-1 + \omega) &= -\frac{1}{\omega^2}(3\zeta(-4)\omega - 2\zeta(-3) + \dots) + \dots \\
S_{-2,-2,1}(-1 + \omega) &= -\frac{1}{\omega^2}(-2S_{-2,1}(\infty) \\
&\quad + \omega(2S_{-3,1}(\infty) + S_{-2,2}(\infty) - \zeta(-2)\zeta(2))) + \dots
\end{aligned}$$

In the above expressions ζ of a negative argument denotes an alternating version of the ζ function:

$$\zeta(-n) \equiv \sum_{k=1}^{\infty} \frac{(-1)^k}{k^n} \quad (21)$$

which is linked to the ordinary ζ function through the well-known relation

$$\zeta(-n) = (2^{1-n} - 1)\zeta(n)$$

The harmonic sums at infinity can be expressed in terms of Euler-Zagier sums. We found the following relations useful: $S_{-2,1}(\infty) = \frac{5}{6}\zeta(-3)$ and $2S_{-3,1}(\infty) + S_{-2,2}(\infty) = -\frac{37}{16}\zeta(4)$. Some of them can be proven, but some we obtained using the program EZ-Face [68]. To simplify the final form we also used that $\zeta(2)^2 = \frac{5}{2}\zeta(4)$. If we plug all these expressions into the wrapping correction

$$128S_1^2[-5\zeta(5) - 4S_{-2}\zeta(3) - 2S_5 + 2S_{-5} + 4S_{4,1} - 4S_{3,-2} + 4S_{-2,-3} - 8S_{-2,-2,1}]$$

we obtain the leading singularities around $M = -1 + \omega$ as

$$\gamma_8^{wrapping}(\omega) \sim 256 \left(\frac{2}{\omega^7} - \frac{8\zeta(2)}{\omega^5} + \frac{9\zeta(3)}{\omega^4} + \frac{59\zeta(4)}{4\omega^3} + \mathcal{O}\left(\frac{1}{\omega^2}\right) \right) \quad (22)$$

which, when combined with the Bethe Ansatz result (11)

$$\gamma_8^{Bethe}(\omega) \sim 256 \left(\frac{-2}{\omega^7} + \frac{0}{\omega^6} + \frac{8\zeta(2)}{\omega^5} - \frac{13\zeta(3)}{\omega^4} - \frac{16\zeta(4)}{\omega^3} + \mathcal{O}\left(\frac{1}{\omega^2}\right) \right) \quad (23)$$

agrees with LO and NLO BFKL expectations

$$\gamma_8(\omega) \sim -256 \left(\frac{4\zeta(3)}{\omega^4} + \frac{\frac{5}{4}\zeta(4)}{\omega^3} + \mathcal{O}\left(\frac{1}{\omega^2}\right) \right) \quad (24)$$

Finally we note that we can continue around any other negative integers to compare with other predictions. In particular, it is known that there is an ω^{-7} pole at even negative integers [35], whose coefficient is fully reproduced just by the Bethe Ansatz answer. We will thus check that our wrapping correction does not give any contribution to this pole which we analyze at $M = -2 + \omega$. The terms which can contribute at this order are S_5 and S_{-5} . Since the analytical continuations are $S_{\pm 5}(M = -2 + \omega) = -\frac{1}{\omega^5}$ there is no contribution from wrapping at this order which is consistent with the BA results.

9 Conclusions

In this paper, using integrability properties of the light-cone quantized world-sheet QFT of the string in $AdS_5 \times S^5$, we have obtained the four-loop wrapping correction to the anomalous dimension of twist-2 operators with spin M . As a first check we determined the large M asymptotic behaviour of our expression and concluded that it does not modify the Bethe Ansatz result for the cusp anomalous dimension. In contrast, the wrapping correction is essential for the correct behaviour of the twist-2 anomalous dimensions under analytic continuation to $M = -1 + \omega$. Indeed it exactly cancels all the higher order poles in the Bethe Ansatz result and the remaining leading poles exactly agree with LO and NLO BFKL predictions.

Let us note that despite the apparent complexity of the computation, the string theory calculation performed in the present paper is still much simpler than any corresponding gauge theory perturbative computation. This

suggests that one can use string theory methods of the AdS/CFT correspondence as an efficient calculational tool even in the weak coupling perturbative regime.

In fact, it would be interesting to analyze in detail the interrelations between the perturbative and string theory computations as it might give a hint on streamlining higher loop perturbative methods.

Moreover it would be very interesting to use the complete formulas for the 4-loop anomalous dimensions to understand better the underlying theoretical structure both from the point of view of their asymptotic properties along the lines of [58, 59] and of the analyticity structure for higher negative integers as discussed in [35] and further aspects of the link with BFKL like in [70].

Apart from these physics issues, there still remain several aspects of our derivation which may be improved. We have managed to do the exact calculation of the result for the terms proportional to the $\zeta(3)$ and $\zeta(5)$ (see Appendix B) but the derivation of the rational part was essentially based on the maximum transcendentality conjecture. It would be very instructive to obtain this result from first principles.

Finally let us note that the exact form of the Lüscher corrections may be a precision test that has to be satisfied by any yet-to-be constructed exact spectral equation (or more probably a set of coupled nonlinear integral equations). Our computation in this paper and the very nontrivial consistency requirement with the analytical structure predicted from BFKL can be understood as a check of the form of the Lüscher corrections for the worldsheet QFT of the $AdS_5 \times S^5$ superstring.

Acknowledgments. We thank Vitaly Velizhanin and the authors of [35] for sharing with us formula (11). This work has been supported in part by Polish Ministry of Science and Information Technologies grant 1P03B04029 (2005-2008), RTN network ENRAGE MRTN-CT-2004-005616, ToK grant COCOS MTKD-CT-2004-517186. ZB thanks the Jagellonian University for warm hospitality during the time when this work was performed and for the Marie Curie ToK COCOS grant for the financial support. ZB was also supported by a Bolyai Scholarship and OTKA 60040.

A Evaluation of the integrand

In this appendix we elaborate on the super-trace part of the integrand. Since we expect the wrapping contribution to appear at order g^8 the super-trace part has to vanish at leading order. We check this requirement in the first subsection and then make a systematical small g^2 expansion of each term in the second one.

Zeroth order contribution

Since the rapidities appear always in pairs $(u, -u)$ we found it useful to combine their contributions. Using the explicit form of these functions from [28] (see Appendix D below) we evaluate the expressions entering the evaluation of the supertrace

$$SB0(q, u)SB0(q, -u) = 1$$

for any g . To abbreviate the formulas we introduce the following notation

$$f(g) = f_0(1 + g^2 \delta f + O(g^4))$$

In this notation the single diagonal bosonic part is given by

$$SBQ(q, u)_0 SBQ(q, -u)_0 = \frac{(q + i(Q - 1))^2 - 4u^2}{(q - i(Q - 1))^2 - 4u^2}$$

While the diagonal fermionic part for $j = 0, \dots, Q - 1$ reads as

$$SF(q, u, j)_0 SF(q, -u, j)_0 = \frac{(q - i(Q - 1) + 2ij)^2 - 4u^2}{(q - i(Q - 1))^2 - 4u^2}$$

The main complication is in the matrix part. One can observe, however, that evaluating the matrix part at $g = 0$ the resulting matrix $SB(q, u, j)_0$ has eigenvectors $(\frac{4i(Q-j)}{(q+iQ)^2}, 1)$ and $(\frac{-4ij}{q^2+Q^2}, 1)$, which are independent of the u_k . Thus we can diagonalize all of them simultaneously leading to

$$G SB(q, u, j)_0 G^{-1} = \begin{pmatrix} \frac{i+2ij+q-iQ-2u}{i+q-iQ-2u} & 0 \\ 0 & \frac{2ij+q-i(1+Q)-2u}{i+q-iQ-2u} \frac{2u-i}{2u+i} \end{pmatrix}$$

Let us denote the corresponding eigenvalues by $SB^\pm(q, u, j)_0$. As a consequence

$$\text{Tr} \left(\prod_{k=1}^M SB(q, u_k, j) \right)_0 = \prod_{k=1}^M SB^+(q, u_k, j)_0 + \prod_{k=1}^M SB^-(q, u_k, j)_0$$

where we have

$$SB^\pm(q, u, j)_0 SB^\pm(q, -u, j)_0 = \frac{(q - i(Q \mp 1) + 2ij)^2 - 4u^2}{(q - i(Q - 1))^2 - 4u^2}$$

Now couple of observations are in order. We can see that

$$SB^+(q, u, 0)_0 SB^+(q, -u, 0)_0 = SB0(q, u) SB0(q, -u) = 1$$

and that

$$SBQ(q, u)_0 SBQ(q, -u)_0 = SB^-(q, u, Q)_0 SB^-(q, -u, Q)_0$$

This means that we can incorporate the contribution of $SB0$ into SB^+ by extending the summation over j in (16) from $j = 1$ to $j = 0$. In a similar manner, the summation of the contributions of SB^- when extended naively to $j = Q$ turns out to automatically incorporate the contribution of SBQ . We can further realize that

$$SB^+(q, u, j)_0 SB^+(q, -u, j)_0 = SB^-(q, u, j+1)_0 SB^-(q, -u, j+1)_0$$

Thus the two bosonic eigenvalues contribute in the same way. This means that at leading order the super-trace part reads as

$$\text{STr} \left(\prod_{k=1}^M S_{Q_{matrix}}^{sl(2)}(q, u_k) \right)_0 = -2 \sum_{j=0}^{Q-1} \prod_{k=1}^M SF(q, u_k, j)_0 + 2 \sum_{j=0}^{Q-1} \prod_{k=1}^M SB^+(q, u_k, j)_0$$

But finally we can observe that the bosonic and the fermionic contributions are the same for each j :

$$SB^+(q, u, j)_0 SB^+(q, -u, j)_0 = SF(q, u, j)_0 SF(q, -u, j)_0$$

so the super-trace part vanishes at leading order and consequently the wrapping contribution also vanishes at order g^4 .

First order contribution

We expand now each function to the order g^2 as follows: The expansion of $SB0$ is trivial, the correction vanishes. The other diagonal bosonic contribution reads as

$$\prod_{k=1}^M SBQ(q, u_k) = \left(\prod_{k=1}^M SBQ(q, u_k)_0 \right) \left[1 + g^2 \sum_{k=1}^M \delta SBQ(q, u_k) \right]$$

The fermionic is given by

$$\prod_{k=1}^M SF(q, u_k, j) = \left(\prod_{k=1}^M SF(q, u_k, j)_0 \right) \left[1 + g^2 \sum_{k=1}^M \delta SF(q, u_k, j) \right]$$

In the matrix case we use the same matrices G we used to diagonalize $SB(q, u, j)_0$ and bring $G SB(q, u, j) G^{-1}$ into the form

$$\begin{pmatrix} SB^+(q, u, j)_0(1 + g^2 \delta SB^+(q, u, j)) & O(g^2) \\ O(g^2) & SB^-(q, u, j)_0(1 + g^2 \delta SB^-(q, u, j)) \end{pmatrix}$$

The advantage of this form is that

$$\begin{aligned} \text{Tr} \left(\prod_{k=1}^M SB(q, u_k, j) \right) &= \left(\prod_{k=1}^M SB^+(q, u_k, j)_0 \right) \left[1 + g^2 \sum_{k=1}^M \delta SB^+(q, u_k, j) \right] + \\ &\quad \left(\prod_{k=1}^M SB^-(q, u_k, j)_0 \right) \left[1 + g^2 \sum_{k=1}^M \delta SB^-(q, u_k, j) \right] \end{aligned}$$

We have checked again by explicit calculation that

$$\delta SB^+(q, u, 0) + \delta SB^+(q, -u, 0) = 0$$

and that

$$\delta SBQ(q, u) + \delta SBQ(q, -u) = \delta SB^-(q, u, Q) + \delta SB^-(q, -u, Q)$$

Thus both bosonic summations can be extended as before. Shifting the summation in the SB^- case we can bring the whole leading correction into the form:

$$\text{STr} \left(\prod_{k=1}^M S_{Q_{matrix}}^{sl(2)}(q, u_k) \right) = g^2 \sum_{j=0}^{Q-1} \left(\prod_{k=1}^M SB^+(q, u_k, j)_0 \right) \sum_{k=1}^M \delta SBF(q, u_k, j)$$

where

$$\delta SBF(q, u_k, j) = \delta SB^+(q, u_k, j) + \delta SB^-(q, u_k, j+1) - 2\delta SF(q, u_k, j)$$

We could explicitly calculate this quantity, which turned out to be

$$\delta SBF(q, u_k, j) = \frac{16}{1 + 4u_k^2} \left[\frac{1}{2j - iq - Q} - \frac{1}{2(j+1) - iq - Q} \right]$$

B Exact calculation of the coefficient of $\zeta(3)$ and $\zeta(5)$

Having taken the residue at $q = iQ$ we will encounter derivatives of the Baxter's Q functions (denoted here by $P_M(\cdot)$ to avoid confusion with Q) around four different points, namely around $\pm \frac{i}{2}$ and $\pm \frac{i}{2} + iQ$. In the following calculation it is more convenient to use the function with a shifted argument

$$U(\pm Q) = P_M(\pm \frac{i}{2} + iQ)$$

It has the following expansion around $Q = 0$:

$$U(Q) = 1 + 2S_1 Q + 2(S_1^2 + S_{-2}) Q^2 + \dots$$

After taking the residue of the integrand we have a sum of rational functions of Q where the denominators are products of powers of Q and $U(Q)U(-Q)$. We can make a partial fraction expansion of this result and separate the terms having denominator Q^n :

$$\frac{A_5}{Q^5} + \frac{A_4}{Q^4} + \frac{A_3}{Q^3} + \frac{A_2}{Q^2} + \frac{A_1}{Q} + \sum_{i,j=1}^{i+j \leq 5} \frac{A_{ij}(Q)}{U(Q)^i U(-Q)^j}$$

Explicit calculation showed that the quantities of interest read as

$$A_5 = 10 \quad ; \quad A_3 = U''(0) - 4U'(0)^2 + 12(\partial_Q \tilde{T}(q, Q))|_{q=iQ=0}$$

In writing A_3 into the form above we used $U(0) = 1$ and the previously calculated values of A_5 and $A_4 = 12(\tilde{T}(0, 0) + U'(0)) = 0$. The other coefficients A_2 and A_1 and especially the polynomials $A_{ij}(Q)$ are very cumbersome to compute. Using the expansion of $U(Q)$ in terms of harmonic sums together with $(\partial_Q \tilde{T}(q, Q))|_{q=iQ=0} = S_1^2 - S_{-2}$ we obtain the form of the wrapping correction

$$\Delta E = -128g^8 S_1^2 [5\zeta(5) + 4S_{-2}\zeta(3) + \dots]$$

where the ellipsis denotes a rational number which can be expanded in terms of harmonic sums and which we determine in section 6.

We note that using perturbative methods in the $\mathcal{N} = 4$ super Yang-Mills theory the coefficient of $\zeta(5)$ was calculated in [69]. It is exactly reproduced by our computation. The conjecture for the coefficient of $\zeta(3)$ given in [69] disagrees, however, with our result.

C Analytical continuation of harmonic sums

There is a pronounced difference in the analytical continuation of harmonic sums between the cases when all indexes are positive and when we have at least one negative index. The former case can be described systematically, while the latter one requires a case by case study.

Analytical continuation with only positive indices

The analytical continuation of harmonic sums with all indices positive can be done inductively. One starts with the simplest one $S_a(n)$ and uses the general strategy to move the variable n from the upper bound of the sum to the summand as

$$S_a(n) = \left(\sum_{j=1}^{\infty} - \sum_{j=n+1}^{\infty} \right) \frac{1}{j^a} = S_a(\infty) - \sum_{k=1}^{\infty} \frac{1}{(k+n)^a}$$

Since we are interested in the analytical continuation around -1 we explicitly separate the singular and regular pieces as

$$S_a(-1+x) = -\frac{1}{x^a} + S_a(\infty) - d_a(x)$$

where we found it useful to introduce the function

$$d_{\pm a}(x) = \sum_{k=1}^{\infty} \frac{(\pm 1)^k}{(k+x)^a}$$

This function is regular around $x = 0$ and has the expansion

$$d_a(x) = \zeta(a) - xa\zeta(a+1) + x^2 \binom{a+1}{2} \zeta(a+2) + \dots + (-1)^n x^n \binom{a+n-1}{n} \zeta(a+n) + \dots$$

Suppose now that we have already analytically continued $S_{b,\dots,c}(n)$ and then we want to continue analitically $S_{a,b,\dots,c}(n)$. Using the previous strategy we can write

$$S_{a,b,\dots,c}(n) = \left(\sum_{j=1}^{\infty} - \sum_{j=n+1}^{\infty} \right) \frac{1}{j^a} S_{b,\dots,c}(j) = S_{a,b,\dots,c}(\infty) - \sum_{k=1}^{\infty} \frac{1}{(k+n)^a} S_{b,\dots,c}(k+n)$$

Let us specify this formula for the case of interest, namely to $S_{a,b}(n)$ around $n = -1$. Focusing only on the singular part, which comes from the $k = 1$ term, we have

$$S_{a,b}(-1+x) = -\frac{1}{x^a} [S_b(x)] + \text{reg}$$

where by “reg” we mean terms regular for $x \rightarrow 0$ and

$$S_b(x) = S_b(\infty) - d_b(x) = xb\zeta(b+1) - x^2 \binom{b+1}{2} \zeta(b+2) + \dots$$

This is all we need for the analytical continuation for positive indexes.

Analytical continuation with at least one negative index

In the case when we have at least one negative index in the harmonic sum we have to be careful. The main problem is that the harmonic sum determines two different analytical functions, depending on whether we continue from even or from odd values. Since in our problem we continue from even ones we have to extend our original harmonic sum to odd values first, then to do the analytical continuation. Since the way how one can do this extension for the functions we need is thoroughly explained in [67] we merely cite the result. For instance, in the simplest case when we have just one index the function

$$\bar{S}_{-a}^+(n) = (-1)^n S_{-a}(n) + (1 - (-1)^n) S_{-a}(\infty)$$

is the same as $S_{-a}(n)$ for even values and is extended to odd values in such a way that it can be described by one function. In calculating the analytical continuation around -1 we use the method of moving n from upper bound to the summand and obtain

$$\bar{S}_{-a}^+(-1+x) = \frac{1}{x^a} + S_{-a}(\infty) + d_{-a}(x)$$

where the function $d_{-a}(x)$ is regular around zero and have the expansion:

$$d_{-a}(x) = \zeta(-a) - xa\zeta(-a-1) + x^2 \binom{a+1}{2} \zeta(-a-2) + \dots$$

Similarly one can define the extension of the harmonic sums with the first index being negative to be

$$\bar{S}_{-a,b,\dots,c}^+(n) = (-1)^n S_{-a,b,\dots,c}(n) + (1 - (-1)^n) S_{-a,b,\dots,c}(\infty)$$

as the proper continuation from even values to odd ones. Its analytical continuation around -1 has the singular part

$$\bar{S}_{-a,b,\dots,c}^+(-1+x) = \frac{1}{x^a} [S_{b,\dots,c}(x)] + \text{reg}$$

The analytical continuation of the function $S_{a,-b}(n)$ is more tricky and one has to use the functional relations the harmonic sums satisfy to define

$$\bar{S}_{a,-b}^+(n) = (-1)^n S_{a,-b}(n) + (1 - (-1)^n)(S_{a,-b}(\infty) - S_b(\infty)(S_a(\infty) - S_a(n)))$$

Thus the singular part around -1 reads as

$$\bar{S}_{a,-b}^+(-1+x) = -\frac{1}{x^a}(2S_{-b}(\infty) - \bar{S}_{-b}^+(x)) = -\frac{1}{x^a}(S_{-b}(\infty) + d_{-b}(x))$$

We also need the analytical continuation of $S_{-2,-2,1}(n)$ so we need

$$\bar{S}_{-a,-b,c}^+(n) = \sum_{j=1}^n \frac{1}{j^a} \bar{S}_{-b,c}^+(j) + S_{-b,c}(\infty)(\bar{S}_a^+(n) - S_a(n))$$

Its analytical continuation has a singular part:

$$\bar{S}_{-a,-b,c}^+(-1+x) = -\frac{1}{x^a} \bar{S}_{-b,c}^+(x) + S_{-b,c}(\infty)\left(\frac{1}{x^a} - \frac{-1}{x^a}\right)$$

in which we have

$$\begin{aligned} \bar{S}_{-b,c}^+(x) &= S_{-b,c}(\infty) - \sum_{k=1}^{\infty} \frac{(-1)^k}{(k+x)^b} S_c(k+x) \\ &= x(bS_{-b-1,c}(\infty) + cS_{-b,c+1}(\infty) - cS_{-b}(\infty)S_{c+1}(\infty)) + \dots \end{aligned}$$

D Explicit S-matrix coefficients

In this appendix we collect the explicit expressions for the matrix part of the $Q-1$ scattering matrix derived in [28]. We choose the normalization as

$$SB0(q, u) = a_1^1(q, u) = 1$$

The bosonic matrix part reads as

$$SB(q, u, j) = \begin{pmatrix} \frac{Q+1-j}{Q+1}a_1^1(q, u) + \frac{j}{2}a_2^2(q, u) & \frac{j(Q-j)}{Q-1}a_4^2(q, u) \\ \frac{1}{Q-1}a_2^4(q, u) & \frac{2(Q-j)}{(Q-1)^2}a_4^4(q, u) + \frac{j-1}{2}a_{10}^{10}(q, u) \end{pmatrix}$$

where $j = 1, \dots, Q-1$. The last bosonic part is simply

$$SQ(q, u) = \frac{1}{Q+1}a_1^1(q, u) + \frac{Q}{2}a_2^2(q, u)$$

while the fermionic part is given by

$$SF(q, u, j) = \frac{Q-j}{Q^2}a_6^6(q, u) + \frac{j}{2}a_7^7(q, u)$$

The various matrix elements read in terms of $x^\pm(u)$ and $z^\pm(q)$ as

$$\begin{aligned} a_6^6 &= Q \frac{x^- - z^-}{x^+ - z^-} \sqrt{\frac{x^+}{x^-}} \quad ; \quad a_7^7 = \frac{2}{Q} \frac{z^-(x^- - z^+)(1 - x^- z^+)}{z^+(x^+ - z^-)(1 - x^- z^-)} \sqrt{\frac{x^+}{x^-}} \\ a_{10}^{10} &= \frac{2}{Q-1} \frac{z^-(x^- - z^+)(1 - x^+ z^+)}{z^+(x^+ - z^-)(1 - x^- z^-)} \\ a_4^2 &= -i \frac{Q-1}{Q} \frac{z^-(x^- - x^+)}{z^+(x^+ - z^-)(1 - x^- z^-)} \quad ; \quad a_2^4 = i \frac{Q-1}{Q} \frac{(z^- - z^+)^2(x^- - x^+)}{(x^+ - z^-)(1 - x^- z^-)} \\ a_2^2 &= -\frac{1}{Q(1+Q)} \frac{1}{z^+(x^+ - z^-)(1 - x^- z^-)} \left[2z^- z^+ (Q + x^- z^- - (1+Q)x^- z^+) \right. \\ &\quad \left. + 2x^+ (z^+ + z^-(-1 + Q(-1 + x^- z^+))) \right] \\ a_4^4 &= -\frac{(Q-1)}{2Qx^+x^-(x^+ - z^-)(1 - x^- z^-)} \left[x^-(Q(x^-)^2 x^+ z^- - x^-(x^+ + z^-) \right. \\ &\quad \left. + x^+ z^-(2 - x^+ z^-)) - (x^- - x^+)x^+ z^-(z^- - z^+) \right] \end{aligned}$$

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